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Expansions for the Roots of Polynomials which Contain Parameters

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Methods are presented for calculating the perturbation series for the roots of polynomials which contain parameters in addition to the perturbation parameter. In particular the complicating effects of multiplicities of the unperturbed roots are examined.

PRELIMINARIES

Consider the polynomial equation of degree n in λ,

$$f(\lambda, \alpha, \xi) = \lambda^n + a_{n-1}(\alpha, \xi)\,\lambda^{n-1} + \cdots + a_0(\alpha, \xi) = 0, \tag{1}$$

where the coefficients $a_i(\alpha, \xi)$ are functions of a set of one or more parameters α and are analytic in the perturbation parameter ξ at $\xi = 0$. If $n \leqslant 4$ the roots $\lambda(\alpha, \xi)$ of Eq. (1) can, in principle, be obtained in global form as functions of α and ξ. If $n > 4$, however, this is not, in general, possible and may not be practical even when possible in principle. It is sometimes useful to approximate the roots in the form of truncated power series in $\xi^{1/q}$, where q is a positive integer, the values of which must be determined. Although there are standard methods for obtaining such series the presence of the variable parameters α creates problems that require special attention. A major difficulty results from the fact that the required values of q can depend upon the values of the set α. If different values of q apply in two different regions of α, the two series do not, in general, approach one another as α varies continuously from one region to the other. Rather, the radius of convergence of one series approaches zero, and the completely new series becomes applicable as the boundary is crossed.

It is ordinarily assumed that the polynomial $f(\lambda, \alpha, \xi)$ is irreducible, i.e., that is cannot be factored into two or more polynomials with coefficients analytic in ξ. If the polynomial is reducible, one merely treats its irreducible factors separately. The irreducibility, however, can depend upon the values of

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the α, and the forms of the series for the various roots can change abruptly is going from a region of α in which $f(\lambda, \alpha, \xi)$ is irreducible to a region in which $f(\lambda, \alpha, \xi)$ is reducible. There are phenomena other than factoring that can cause the forms of the series to change. The types of possible behavior are so numerous that it is not practical to consider most of them in this article. Fortunately, a few simple cases occur frequently in applications. Although the treatment here is primarily concerned with these special cases, some idea of how more complicated behaviors can be treated is obtained. After a brief discussion of a few preliminary general questions in the present section we shall consider special cases in subsequent sections.

At values of α and ξ for which the roots are all simple there is no problem in principle, since the value of q will be unity. The conventional methods used for polynomials with numerical coefficients apply to these cases without theoretical complications. It is the combined presence of multiple roots and the parameters α that causes the new difficulties.

One would like to know the regions of α over which $f(\lambda, \alpha, \xi)$ is irreducible. Unfortunately, there is no generally applicable test for irreducibility. A necessary and sufficient condition that $f(\lambda, \alpha, \xi)$ have a *repeated* factor is that its discriminant $\mathrm{D}(\alpha, \xi)$ be identically zero as a function of ξ. The condition $\mathrm{D}(\alpha, \xi) \equiv 0$ is, therefore, *sufficient* but *not necessary* for $f(\lambda, \alpha, \xi)$ to be reducible and is not often useful as a test for reducibility. The condition $\mathrm{D}(\alpha, 0) = 0$ is necessary and sufficient for $f(\lambda, \alpha, 0) = 0$ to have at least one multiple root.

Suppose that throughout a certain region of α the polynomial $f(\lambda, \alpha, \xi)$ has a root of multiplicity m at $\xi = 0$ but is irreducible for all $\xi \neq 0$ in a ξ-neighborhood about $\xi = 0$. We focus our attention on these m roots that are equal at $\xi = 0$ and designate them $\lambda_i(\alpha, \xi)$, where $i = 1, 2, \dots, m$. All of these must be different functions of ξ in the ξ-neighborhood, except at $\xi = 0$. This follows directly from a theorem on algebraic functions [1]. In the present context it is convenient to express this result in the form of

THEOREM I. *If a perturbation renders a polynomial irreducible all multiplicities of the unperturbed polynomial are removed by the perturbation.*

Thus, if two roots are identical functions of ξ, they must belong to separate factors of a reducible polynomial.

When $f(\lambda, \alpha, \xi)$ is irreducible the m roots under discussion belong to one or more cycles [2]. The number of roots in a particular cycle is the value of q for that cycle, and $\sum q = m$, where the sum is over the different cycles. For a particular cycle the series for the q roots take the form

$$\lambda(\alpha, \xi) = \lambda_0(\alpha) + \sum_{j=1} \lambda_j(\alpha)\, \xi^{j/q}, \tag{2}$$

where $\lambda_0(\alpha)$ is the common value of all the m roots at $\xi = 0$, and the q different roots are obtained by using the q different qth roots of ξ, all with the same coefficients $\lambda_j(\alpha)$. Or, if we number the roots of the cycle, $i_1, \ldots, i_q$, we can express the roots by

$$\lambda_{i_k}(\alpha, \xi) = \lambda_0(\alpha) + \sum_{j=1} \lambda_j(\alpha)\, \omega^{k-1} \xi^{j/q} \qquad (k = 1, \ldots, q), \tag{3}$$

where $\xi^{1/q}$ is the real positive qth root, and ω is a primitive qth root of unity

$$\omega = e^{2\pi i/q} \qquad (i = \sqrt{-1}). \tag{4}$$

The $\lambda_j(\alpha)$ have the same values for all the roots of the cycle.

In many applications it is known that the roots $\lambda(\alpha, \xi)$ must be real for real ξ. For example, the roots are often the eigenvalues of a real symmetric matrix or of a Hermitian matrix. For such cases we have

THEOREM II. *If a perturbation renders a polynomial irreducible, and the roots must be real for all real positive values of the perturbation parameter, then no cycle can have a degree (value of q) greater than two.*

This follows directly from Eq. (2), since only square roots are all real. We also have

THEOREM III. *If a perturbation renders a polynomial irreducible, and the roots must be real for all real positive and negative values of the perturbation parameter, then no cycle can have a degree greater than one.*

This follows from Theorem II and the fact that $\xi^{1/2}$ and $(-\xi)^{1/2}$ cannot both be real. This theorem also follows directly from Rellich's theorem [3].

We digress here briefly to point out, in connection with applications of the last two theorems, the following possible difference between a matrix formulation and the polynomial formulation of a perturbation problem. Suppose that the matrix formulation is $\mathbf{H} = \mathbf{A} + \beta\mathbf{V}$, where the matrix $\mathbf{H}$ is Hermitian and the perturbation parameter β can take on both negative and positive real values. It often happens that both $\mathbf{A}$ and $\mathbf{V}$ are traceless in such a way that in the characteristic polynomial only even powers of β occur. In the polynomial formulation the "natural" perturbation parameter would then be a positive-only parameter ξ, e.g., $\xi = \beta^2$. Theorem III would apply in the matrix formulation in terms of β, while Theorem II would apply to the polynomial form in terms of ξ. The two formulations must, of course, lead to equivalent results.

We return now to our consideration of the m roots $\lambda_i(\alpha, \xi)$, $i = 1, \ldots, m$, which, when unperturbed are equal. Information about their cycle structure can be obtained from expressions for the first nonzero coefficients $\lambda_j(\alpha)\omega^{k-1}$ in Eq. (3).

The possible structures are, of course, very numerous. We must restrict our treatment to a few cases which occur very frequently in physical applications. To conveniently classify these cases we express $f(\lambda, \alpha, \xi)$ in the form,

$$f(\lambda, \alpha, \xi) = f(\lambda, \alpha, 0) + \xi^p \phi(\lambda, \alpha, \xi), \tag{5}$$

where

$$f(\lambda, \alpha, 0) = [\lambda - \lambda_0(\alpha)]^m \prod_{j=m+1}^{n} [\lambda - \lambda_j(\alpha, 0)]; \tag{6}$$

$\lambda_j(\alpha, 0)$ are the other unperturbed roots of the equation, and $p \geqslant 1$ is the highest power of ξ that can be factored from the perturbation part of $f(\lambda, \alpha, \xi)$. Form (5) must exist locally since the coefficients $a_i(\alpha, \xi)$ in Eq. (1) are analytic in ξ and can, therefore, be expanded in an ordinary power series in ξ.

To investigate the first nonzero coefficients we apply Newton's polygon method [4] to the equation in the form given by Eq. (5).

The First Nonzero Coefficient and Cycle Structure

Our first step in the classification of the cases to be considered here is to again focus our attention on the m roots which have the common value $\lambda_0(\alpha)$ and to classify these roots according to whether $\lambda_0(\alpha)$ *is not* or *is* a root of $\phi(\lambda, \alpha, 0)$. This is the same as classifying them according to whether $\phi[\lambda_0(\alpha), \alpha, 0]$ is *not zero* or *is zero.* We designate these Cases I and II, respectively, and consider them in that order, the first being the simpler case.

Case I. $\lambda_0(\alpha)$ is not a root of $\phi(\lambda, \alpha, 0)$.

If we apply Newton's polygon to this problem we find that the only side involved is the one through the points $(0, m)$ and $(p, 0)$. This corresponds to the substitutions

$$\xi = t^m \qquad \text{and} \qquad \lambda(\alpha, t) = \lambda_0(\alpha) + u(\alpha, t)t^p. \tag{7}$$

Standard procedure [4] then leads to the series

$$\lambda(\alpha, \xi) = \lambda_0(\alpha) + \lambda_p(\alpha)\, \xi^{p/m} + \lambda_{p+1}(\alpha)\, \xi^{(p+1)/m} + \cdots, \tag{8}$$

with

$$\lambda_p(\alpha) = \left[-\phi[\lambda_0(\alpha), \alpha, 0] \Big/ \prod_{j=m+1}^{n} [\lambda_0(\alpha) - \lambda_j(\alpha, 0)]\right]^{1/m}. \tag{9}$$

Clearly, in this case there can be only one cycle, which is of degree m.

Case II. $\lambda_0(\alpha)$ is a root of $\phi(\lambda, \alpha, 0)$.

When $\lambda_0(\alpha)$ is a root of $\phi(\lambda, \alpha, 0)$ it is necessary to further specify the properties of $\phi(\lambda. \alpha, \xi)$. Suppose the multiplicity of $\lambda_0(\alpha)$ in $\phi(\lambda, \alpha, 0)$ is m_1. $\phi(\lambda, \alpha, \xi)$ can then be expressed as

$$\phi(\lambda, \alpha, \xi) = [\lambda - \lambda_0(\alpha)]^{m_1} H(\lambda, \alpha) + \xi^{p_1} \phi_1(\lambda, \alpha, \xi), \tag{10}$$

where $H(\lambda, \alpha)$ is not a function of ξ, and $p_1 \geqslant 1$.

We shall not consider the cases here in which $\lambda_0(\alpha)$ is a root of $\phi_1(\lambda, \alpha, 0)$. With this restriction subclassification can be made by use of a quantity Q defined by

$$Q = m_1 p + m_1 p_1 - m p_1. \tag{11}$$

The three cases $Q > 0$, $Q = 0$, and $Q < 0$ are then to be considered.

A $Q > 0$

One side of Newton's polygon passing through the two points $(0, m)$ and $(p + p_1, 0)$ is involved in this case. This corresponds to the substitutions

$$\xi = t^m \qquad \text{and} \qquad \lambda(\alpha, t) = \lambda_0(\alpha) + u(\alpha, t) t^{p+p_1}, \tag{12}$$

and leads to the result

$$\begin{aligned} \lambda(\alpha, \xi) = \lambda_0(\alpha) &+ \lambda_{p+p_1}(\alpha) \xi^{(p+p_1)/m} \\ &+ \lambda_{p+p_1+1}(\alpha) \xi^{(p+p_1+1)/m} + \cdots, \end{aligned} \tag{13}$$

where

$$\lambda_{p+p_1}(\alpha) = \left[-\phi_1[\lambda_0(\alpha), \alpha, 0] \Big/ \prod_{j=m+1}^{n} [\lambda_0(\alpha) - \lambda_j(\alpha, 0)] \right]^{1/m}. \tag{14}$$

These results are equivalent to Eqs. (8) and (9) with p replaced by $p + p_1$ and ϕ replaced by ϕ_1. It follows that the structure is one cycle of degree m.

B $Q = 0$

Again, only one side of the polygon is involved, but this time the side passes through three points, $(0, m)$, (p, m_1), and $(p + p_1, 0)$. This again corresponds to the substitutions given by Eq. (12), but in this case $\lambda_{(p+p_1)}(\alpha)$ are the solutions of the following equation in u:

$$u^m \prod_{j=m+1}^{n} [\lambda_0(\alpha) - \lambda_j(\alpha, 0)] + u^{m_1} H[\lambda_0(\alpha), \alpha] + \phi_1[\lambda_0(\alpha), \alpha, 0] = 0. \tag{15}$$

If Eq. (15) does not have multiple roots this leads to m different expressions for $\lambda_{(p+p_1)}(\alpha)$. If Eq. (15) has multiple roots one can proceed by use of a second polygon [4]. In either case further specification of the polynomial is required for a delineation of the cycle structures. Since the possibilities are numerous we shall not pursue a general investigation of this case.

$C\ Q<0$

Two sides of the polygon must be used; one through $(0, m)$ and (p, m_1) and the other through (p, m_1) and $(p+p_1, 0)$. These sides correspond, respectively, to the substitutions

$$\xi = t^{m-m_1} \qquad \text{and} \qquad \lambda(\alpha, t) = \lambda_0(\alpha) + u(\alpha, t)t^p, \tag{16}$$

and

$$\xi = t^{m_1} \qquad \text{and} \qquad \lambda(\alpha, t) = \lambda_0(\alpha) + u(\alpha, t)t^{p_1}. \tag{17}$$

Substitutions (16) lead to

$$\lambda(\alpha, \xi) = \lambda_0(\alpha) + \lambda_p(\alpha)\xi^{p/(m-m_1)} + \lambda_{p+1}(\alpha)\xi^{(p+1)/(m-m_1)} + \cdots, \tag{18}$$

with

$$\lambda_p(\alpha) = \left[-H[\lambda_0(\alpha), \alpha] \Big/ \prod_{j=m+1}^{n} [\lambda_0(\alpha) - \lambda_j(\alpha, 0)]\right]^{1/(m-m_1)}. \tag{19}$$

Substitutions (19) gives

$$\lambda(\alpha, \xi) = \lambda_0(\alpha, 0) + \lambda_{p_1}(\alpha)\xi^{p_1/m_1} + \lambda_{p_1+1}(\alpha)\xi^{(p_1+1)/m_1} + \cdots, \tag{20}$$

where

$$\lambda_{p_1}(\alpha) = [-\phi_1[\lambda_0(\alpha), \alpha, 0]/H[\lambda_0(\alpha), \alpha]]^{1/m_1}. \tag{21}$$

These represent two cycles, one of degree $m-m_1$ and one of degree m_1. If the roots must be real for positive ξ we have from Theorem II

$$m - m_1 \leqslant 2, \qquad m_1 \leqslant 2. \tag{22}$$

If the roots must be real for both positive and negative ξ, Theorem III gives

$$m - m_1 \leqslant 1, \qquad m_1 \leqslant 1. \tag{23}$$

THE COEFFICIENTS

The substitutions of the type given by Eqs. (7), (12),(16), and (17) lead to expansions of the form

$$\lambda(\alpha, t) = \lambda_0(\alpha) + \lambda_1(\alpha)t + \lambda_2(\alpha)t^2 + \cdots, \tag{24}$$

where some of the coefficients $\lambda_i(\alpha)$ may be zero. The conventional procedure for obtaining these coefficients is to sustitute the series into Eq. (1) and to set the coefficient of each power of t thus obtained equal to zero. For any particular equation this method is probably as satisfactory as any. For some sufficiently simple, but important, general cases it is possible to obtain general expression for the coefficients. In this section we shall describe one method for doing this.

The method we shall describe is an extension of one presented in an earlier publication [5]. The method previously described cannot be applied directly when multiple roots exist and the parameters α are present.

When there are no multiple roots the procedure [5] is to start with the equation

$$f(\lambda, \alpha, \xi) = \prod_{i}^{n} [\lambda - \lambda_i(\alpha, \xi)], \tag{25}$$

where the roots $\lambda_i(\alpha, \xi)$ are assumed to be expressed in the power series form

$$\lambda_i(\alpha, \xi) = \lambda_{i0}(\alpha) + \lambda_{i1}(\alpha)\xi + \lambda_{i2}(\alpha)\xi^2 + \cdots. \tag{26}$$

The left-hand side of Eq. (25) is given by Eq. (1) or (5). The derivatives of both sides of Eq. (25) with respect to ξ, evaluated at $\xi = 0$ and $\lambda_{i0}(\alpha, 0)$ lead to relationships which give the coefficients $\lambda_{i1}(\alpha)$ in terms of the $\lambda_{i0}(\alpha)$; then the second derivatives lead to $\lambda_{i2}(\alpha)$ in terms of $\lambda_{i0}(\alpha)$ and $\lambda_{i1}(\alpha)$; etc. General expressions have been given for terms through the third power [5].

When there is a root of multiplicity m and for which the appropriate substitution is $\xi = t^q$, mixed derivatives with respect to t and λ are needed to evaluate the coefficients. We use the notation

$$f(\lambda, \alpha, t) = g(\lambda, \alpha, t)\, h(\lambda, \alpha, t), \tag{27}$$

$$g(\lambda, \alpha, t) = \prod_{i=1}^{m} [\lambda - \lambda_i(\alpha, t)], \tag{28}$$

$$h(\lambda, \alpha, t) = \prod_{j=m+1}^{n} [\lambda - \lambda_j(\alpha, t)], \tag{29}$$

and for the various derivatives

$$f^{rs} = \partial^{r+s} f / \partial t^r \, \partial \lambda^s, \tag{30}$$

$$f_i^{rs} = \partial^{r+s} f / \partial t^r \, \partial \lambda^s, \qquad t = 0, \quad \lambda = \lambda_{i0}, \tag{31}$$

and similarly for derivatives of g and h.

The coefficients are obtained in terms of these derivatives and the lower coefficients of all the n roots. The method requires a knowledge of all the n unperturbed roots. It applies whether or not $f(\lambda, \alpha, \xi)$ is irreducible.

In application to a particular problem the procedure is straightforward, but there can be short cuts. For example, for cycles Eq. (3) can be used. Also, the relationships giving the coefficients in Eq. (1) as symmetric functions of the roots can be useful; in particular the relationship

$$\sum_{i=1}^{n} \lambda_{ik}(\alpha) = -(\text{coefficient of } \xi^k \text{ in } a_{n-1}), \tag{32}$$

which holds for all k, is convenient to use.

General expressions for the coefficients are not easy to obtain except for the simplest cases. We give in Tables I and II some coefficients for $m = 2$ and $p = 1$. This case, although simple, is relatively important. Although Theorem II restricts q (not m) to values not greater than 2, in actuality many problems to which the theorem applies do have $m = 2$.

TABLE I

For Reducible or Irreducible $f(\lambda, \alpha, \xi)$, $m = 2$, $p = 1$

$$\lambda_{11} = (-f_1^{11} + \sqrt{(f_1^{11})^2 - h_1^{00} f_1^{20}})/2h_1^{00}$$
$$\lambda_{21} = (-f_1^{11} - \sqrt{(f_1^{11})^2 - 2h_1^{00} f_1^{20}})/2h_1^{00}$$
$$\lambda_{12} = (f_1^{30} + 3\lambda_{11} f_1^{21} + 6\lambda_{11}^2 h_1^{10} - 6\lambda_{11}^2 \lambda_{21} h_1^{01})/6(\lambda_{21} - \lambda_{11}) h_1^{00}$$

λ_{22} is same as λ_{12} with λ_{11} replaced by λ_{21} and vice versa.

$$\begin{aligned} \lambda_{13} = [&f_1^{40} + 4\lambda_{11} f_1^{31} + 12\lambda_{11}^2 h_1^{20} - 24\lambda_{12}(\lambda_{21} - \lambda_{11}) h_1^{10} \\ &- 24\lambda_{11}^2 \lambda_{21} h_1^{11} - 24\lambda_{12}\lambda_{22} h_1^{00} - 24\lambda_{11}(\lambda_{11}\lambda_{22} \\ &+ \lambda_{12}\lambda_{21}) h_1^{01}]/24(\lambda_{21} - \lambda_{11}) h_1^{00} \end{aligned}$$

λ_{23} is same as λ_{13} with λ_{21} replacing λ_{11}, λ_{22} replacing λ_{12}, and vice versa.

$$\begin{aligned} \lambda_{14} = [&f_1^{50} + 5\lambda_{11} f_1^{41} + 20\lambda_{11}^2 h_1^{30} - 60\lambda_{12}(\lambda_{21} - \lambda_{11}) h_1^{20} \\ &- 60\lambda_{11}^2 \lambda_{21} h_1^{21} - 120(\lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{21} - \lambda_{11}\lambda_{13}) h_1^{10} \\ &- 120\lambda_{11}(\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21}) h_1^{11} - 120(\lambda_{12}\lambda_{23} + \lambda_{13}\lambda_{22}) h_1^{00} \\ &- 120\lambda_{11}(\lambda_{11}\lambda_{23} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{21}) h_1^{01}]/120(\lambda_{21} - \lambda_{11}) h_1^{00} \end{aligned}$$

λ_{24} is same as λ_{14} with λ_{21} replacing λ_{11}, λ_{22} replacing λ_{12}, λ_{23} replacing λ_{13}, and vice versa.

TABLE II

For Irreducible $f(\lambda, \alpha, \xi)$, $m = 2$, Cycle of Degree 2, $p = 1$

$\lambda_{11} = -\lambda_{21} = \sqrt{-f_1^{20}/2h_1^{00}}$
$\lambda_{12} = \lambda_{22} = -(f_1^{21} + 2\lambda_{11}^2 h_1^{01})/4h_1^{00}$
$\lambda_{13} = -\lambda_{23} = -(f_1^{40} + 12\lambda_{11}^2 h_1^{20} - 24\lambda_{12}^2 h_1^{00})/48\lambda_{11} h_1^{00}$
$\lambda_{14} = \lambda_{24} = -\{f_1^{41} + 24\lambda_{12} h_1^{20} + 12\lambda_{11}^2 h_1^{21} + 24(2\lambda_{11}\lambda_{13} - \lambda_{12}^2) h_1^{01}\}/48h_1^{00}$

Example

Early in this article it was pointed out that difficulties can occur because cycle structure may be a function of α. The following simple and transparent example illustrates this point:

Consider the following in which α is one parameter,

$$\begin{aligned} f(\lambda, \alpha, \xi) &= \lambda^3 - (\xi + 4)\lambda^2 + (\xi + 5)\lambda + \alpha\xi - 2 \\ &= (\lambda - 1)^2(\lambda - 2) - \xi[(\lambda - 1)\lambda - \alpha] = 0, \end{aligned} \tag{33}$$

where we have

$$\phi(\lambda, \alpha, \xi) = (\lambda - 1)\lambda - \alpha. \tag{34}$$

For $\alpha \neq 0$ this is an example of Case I, $m = 2$, $p = 1$. There is one cycle of degree 2 and one simple root. For the two roots of the cycle one can use Table II, and for the simple root the method described earlier [5] is applicable. The results for $\alpha \neq 0$ are

$$\begin{aligned} \lambda_{1,2} = 1 &\pm \sqrt{\alpha}\,\xi^{1/2} + ((\alpha - 1)/2)\xi \pm ((5\alpha^2 - 10\alpha + 1)/8\sqrt{\alpha})\xi^{3/2} \\ &+ ((2\alpha^2 - 5\alpha + 2)/2)\xi^2 + \cdots, \end{aligned} \tag{35}$$

$$\lambda_3 = 2 + (2 - \alpha)\xi - (2\alpha^2 - 5\alpha + 2)\xi^2 + \cdots. \tag{36}$$

For $\alpha = 0$ we have $\phi(1, \alpha, 0) = 0$. This is not an example of Case II, since $\phi_1 \equiv 0$. Rather, it is clear that these conditions mean that $\lambda - 1$ is a factor of $f(\lambda, 0, \xi)$. This causes the cycle to disappear, and all roots become simple roots. The series for this case are

$$\lambda_1 = 1, \tag{37}$$

$$\lambda_2 = 1 - \xi + 2\xi^2 + \cdots, \tag{38}$$

$$\lambda_3 = 2 + 2\xi - 2\xi^2 + \cdots. \tag{39}$$

In this example the divergence of the series in Eqs. (35) as $\alpha \to 0$ is made clear by the coefficients of $\xi^{3/2}$. This type of difficulty need not always be so obvious.

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